Asymptotic Efficiency of Goodness-of-fit Tests Based on Too-Lin Characterization

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August 24, 2015

Abstract

In this paper a class of new goodness-of-fit tests based on a characterization is presented. The asymptotic properties of test statistics are examined. The tests are compared with some standard tests as well as some tests based on characterizations. For each test the Bahadur efficiencies against some common close alternatives are calculated. A class of locally optimal alternatives is found for each test. The powers of proposed tests for small sample size are obtained.

keywords: `testing of uniformity, moments of order statistics, Bahadur efficiency,` U`-statistics`

MSC(2010): 60F10, 62G10, 62G20, 62G30.

1 Introduction

The uniform distribution is one of the most used distribution in statistical modeling and computer science. Therefore ensuring that the data come from uniform distribution is of huge importance. Moreover, testing that the data come from a particular distribution can be easily reduced to testing uniformity. More about such goodness-of-fit techniques can be found in [6].

In recent times, the tests based on some characteristic property that distribution posesses have become very popular. Many different types of characterizations can be found e.g in [9]. Some characterizations of the uniform distribution can be found e.g in [1], [10], [31]. The first test for uniformity based on characterization that involves moments was proposed by Hashimoto and Shirahata in [12]. Their test was based on Papathanasiou's characterization presented in [25]. Other tests based on characterizations via moments have bee considered in e.g.[16], [17].

One way to compare tests is to calculate their asymptotic efficiencies. In case of non-normal limiting distribution the Bahadur approach to efficiency is suitable. It has been considered in e.g [29], [27], [8], [24], [26].

In this paper we propose new uniformity tests based on characterization that involves the moments of order statistics. We examine the asymptotic properties of the test statistics. This characterization has already been used in [16], however the nature of test is completely different.

The paper is organized as follows. In Section 2 we present the characterization and one class of test statistics based on it. Next, we give the basic of

Bahadur theory. In the Section 3 we state some theorems that will be used for calculations. Using Bahadur efficiency we compare our test with Hashimoto-Shirahata test, Fortiana-Grané test based on maximum correlations (see [8]), as well as some standard goodness-of-fit tests. In Section 4 for each proposed test we found some classes of locally optimal alternatives. Finally, in Section 5 we perform power study and present an application in time series analysis.

2 Characterization and Test Statistics

The following characterization by Too and Lin can be found in [30].

Theorem 2.1 *Let $EX^2_{k,n} < \infty$ for some pair (k,n). Then the equality*

$$EX^2_{k,n} - \frac{2k}{n+1}EX_{k+1,n+1} + \frac{k(k+1)}{(n+1)(n+2)} = 0 \tag{1}$$

holds if and only if $F(x) = x$ on $(0,1)$.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. observations having a continuous distribution function F. Without loss of generality we may assume that the data belong to $[0,1]$ and test for uniform distribution. If we want to test that sample has the distribution function F_0 we may apply our test to transformed sample $F_0(X_1), F_0(X_2), \ldots, F_0(X_n)$.

Denote $X_{(a),X_1,\ldots,X_n}$ the a th order statistic of the sample $X_1, ..., X_n$. In order to test H_0 that $F(x) = x,\ x \in (0,1)$ we propose the following class of test statistics.

$$\begin{aligned} T_n^m = \frac{1}{\binom{n}{m+1}} \sum_{i_1<\cdots<i_{m+1}} \Big(& X^2_{(1),X_{i_1},\ldots,X_{i_m}} - \frac{2}{n+1} X_{(2),X_{i_1},\ldots,X_{i_{m+1}}} \\ & + \frac{2}{(n+1)(n+2)} \Big) \end{aligned}$$

These statistics are asymptotically equivalent to U- statistics with symmetric kernels

$$\begin{aligned} \Phi_m(X_1, .., X_{m+1}) = & \frac{1}{m!} \sum_{\pi(m)} X^2_{(1),X_{\pi_1},\ldots,X_{\pi_m}} - \frac{2}{n+1} X_{(2),X_{i_1},\ldots,X_{i_{m+1}}} \\ & + \frac{2}{(n+1)(n+2)}, \end{aligned}$$

where $\pi(m)$ is the set of all permutations of numbers $1, 2, ..., m$.

The first projections of kernels $\Phi_m(X_1, .., X_{m+1})$ on X_{m+1} under H_0 are

$$\begin{aligned} \phi(s) = E(\Phi_m(X_1, .., X_{m+1})|X_{m+1} = s) = & \frac{m-1}{m} E(X_{(1),s,X_1,\ldots,X_{m-1}}) \\ + \frac{1}{m} E(X_{(1),X_1,\ldots,X_m}) - & \frac{2}{n+1} E(X_{(2),s,X_1,\ldots,X_m}) + \frac{2}{(n+1)(n+2)} \end{aligned}$$

It can be easily shown that they are identically equal to zero under null hypothesis.

The second projections of $\Phi_m(X_1,..,X_{m+1})$ on (X_m, X_{m+1}) under null hypothesis are

$$\begin{aligned}\phi_2^*(s,t) &= E(\Phi_m(X_1,..,X_{m+1})|X_m = s_1, X_{m+1} = t)\\ &= \frac{m-1}{m+1}E(X_{(1),s,t,X_1,...,X_{m-3}}) + \frac{1}{m+1}E(X_{(1),s,X_1,...,X_{m-2}})\\ &+ \frac{1}{m+1}E(X_{(1),t,X_1,...,X_{m-2}}) - \frac{2}{n+1}E(X_{(2),s,t,X_1,...,X_{m-1}})\\ &+ \frac{2}{(n+1)(n+2)}.\end{aligned}$$

After some calculations we obtain

$$\begin{aligned}\phi_m^*(s,t) &= -\frac{2}{m(1+m)^2(2+m)}\Big(-2 + 2(1-t)^m + m^2\big(-(1-s)^{m+1}\\ &+ (1-t)^m t\big) + m\big(-2(1-s)^{m+1} + (1-t)^m + 2(1-t)^m t\big)\Big)\\ &+ \frac{2}{m(1+m)} I\{s<t\}((1-t)^m - (1-s)^m).\end{aligned}$$

Obviously it is not equal to zero. Therefore we conclude that the kernels of our test statistics are weakly degenerate. Hence, we have (see [14])

$$nT_n^m \xrightarrow{d} \binom{m}{2}\sum_{i=1}^{\infty}\nu_i(\tau_i^2 - 1),$$

where ν_i are the eigenvalues of the integral operator S defined by

$$Sf(t) = \int_0^1 \phi_m^*(s,t)f(s)ds.$$

Thus we have to solve the following integral equation

$$\nu f(t) = \int_0^1 f\phi_m^*(s,t)f(s)ds, \tag{2}$$

with constraints $\int_0^1 f(s)ds = 0$ and $\int_0^1 f(t)^2 dt = 1$.

Denote $y(t) = \int_0^t f(s)ds$. After differentiation the expression (2) becomes

$$\nu y''(t) = -2\frac{(1-t)^{m-1}}{m+1}y(t), \quad y(0) = 0, \ \ y(1) = 0, \quad \int_0^1 y'(t)^2 dt = 1.$$

This boundary problem can be solved numerically for particular m.

3 Local Bahadur Efficiency

We choose Bahadur asymptotic efficiency as a measure of the quality of tests. One of the reasons is that the asymptotic distributions of our test statistics are not normal. The Bahadur efficiency can be expressed as the ratio of the Bahadur exact slope, a function describing the rate of exponential decrease for the attained level of significance under the alternative, and the double Kullback-Leibler distance between the null and the alternative distribution. We present brief review of the theory, for more about this topic we refer to [3], [18].

According to Bahadur's theory the exact slopes are defined in the following way. Suppose that under alternative

$$T_n \xrightarrow{P_\theta} b(\theta).$$

Also suppose that the large deviation limit

$$\lim_{n\to\infty} n^{-1} \ln P_{H_0} \left(T_n \geq t\right) = -f(t) \tag{3}$$

exists for any t in an open interval I, on which f is continuous and $\{b(\theta),\, \theta > 0\} \subset I$. Then the Bahadur exact slope is defined as

$$c_T(\theta) = 2f(b(\theta)). \tag{4}$$

The Raghavachari's inequality

$$c_T(\theta) \leq 2K(\theta),\, \theta > 0, \tag{5}$$

where $K(\theta)$ is the Kullback-Leibler distance between the alternative H_1 and the null hypothesis H_0, leads to natural definition of Bahadur efficiency. Since it is very important for a test to distinguish close alternatives from null distribution, we define local Bahadur efficiency as

$$e^B(T) = \lim_{\theta\to 0} \frac{c_T(\theta)}{2K(\theta)}. \tag{6}$$

Suppose that the alternative density functions satisfy the regularity conditions from ([18], Chapter 6). Denote $h(x) = \frac{\partial g(x,\theta)}{\partial \theta}|_{\theta=0}$. Then it can be shown that the double Kullbac-Leibler distance between the null distribution and the close alternative can be expressed as

$$2K(\theta) = I(g)\theta^2 + o(\theta^2), \quad \theta \to 0, \tag{7}$$

where $I(g) \in (0, \infty)$ is the Fisher information function

$$I(g) = \int_0^1 \frac{h^2(x)}{g(x,0)} dx.$$

In order to find Bahadur exact slopes we use the following theorems from [23] and [22] for $U-$ and $V-$ statistics with weakly degenerate kernels. The first theorem gives large deviation function f and the second the limit in probability $b(\theta)$ that appear in (4).

Theorem 3.1 *Let the kernel Φ of $U-$ statistic be bounded and weakly degenerate. Let λ_0 be the smallest λ satisfying the integral equation*

$$x(s_1) = \lambda \int_0^1 \Phi^*(s_1, s_2)x(s_2)ds_2, \tag{8}$$

and suppose that λ_0 is a simple characteristic number of the linear integral operator with kernel Φ^* acting from $L^2[0,1]$ into $L^2[0,1]$. Then, for sufficiently small $a > 0$, we have

$$\lim_{n\to} n^{-1} \log P\{T_n \geq a\} = \sum_{j=2}^{n} c_j a^{\frac{j}{2}},$$

where the right-hand side is a convergent series with respect to a. Moreover

$$c_2 = -\frac{\lambda_0}{m(m-1)}.$$

Theorem 3.2 *Suppose that the family of alternatives $\mathcal{G}$ satisfy some regularity conditions from [22]. Than*

$$b(\theta) = \frac{m(m-1)}{2} \int_{R^2} \phi^*(x_1, x_2) g_\theta(x_1, 0) g_\theta(x_1, 0) dx_1 dx_2 \cdot \theta^2 + o(\theta^2), \quad \theta \to 0. \tag{9}$$

We now calculate the local Bahadur efficiencies for test statistics T_n^1 and T_n^2. The tests with greater values of m have little or no practical value.

The second projections of their kernels under H_0 are respectively

$$\phi_1^*(s_1, s_2) = \frac{s_1^2}{2} + \frac{s_2^2}{2} - \max(s_1, s_2) + \frac{1}{3}, \tag{10}$$

$$\begin{aligned} \phi_2^*(s_1, s_2) = \frac{1}{3} I\{s < t\}(s^2 - 2t + t^2 - s^2) + \frac{1}{3} I\{s > t\}(t^2 - 2s + s^2 - t^2\Big) + \frac{1}{6} \\ + \frac{1}{3}s^2 + \frac{1}{3}t^2 - \frac{2}{9}s^3 - \frac{2}{9}t^3. \end{aligned} \tag{11}$$

Putting $\lambda = \nu^{-1}$, in case of (10), the expression (2) becomes well-known Sturm-Liouville equation

$$y''(t) + \lambda y(t) = 0, \ y(0) = 0, \ y(1) = 0.$$

We obtain the eigenvalues $\lambda_k = k^2\pi^2, \ \ k = 1, 2...$ Therefore the main characteristic value is $\lambda_1 = \pi^2$. It is interesting to note that Cramer-von Mises statistic has the same kernel. Therefore they have similar asymptotic properties.

Putting $\lambda = \nu^{-1}$, in case of (11), the expression (2) becomes

$$y''(t) + \frac{2}{3}(1-t)y(t) = 0, \ y(0) = 0, \ y(1) = 0.$$

Using numerical methods we obtain that the main characteristic value is $\lambda_1 = 28.4344$.

3.1 Competitor tests

We compare our tests with the following tests

- Kolmogorov-Smirnov test with test statistic

$$D_n = \sup_{t\in(0,1)} |F_n(t) - t|; \tag{12}$$

- Anderson-Darling test with test statistic

$$A_n^2 = \int_0^1 \frac{(F_n(t)-t)^2}{t(1-t)} dt; \tag{13}$$

- Cramer-von Mises test with test statistic

$$W_n^2 = \int_0^1 (F_n(t)-t)^2 dt; \tag{14}$$

- the test based on maximum correlations [**?**] with test statistic

$$Q^c = |Q_n - 1| = |\frac{6}{n^2}\sum_{i=1}^{n}(2i-n-1)X_{(i)} - 1|; \tag{15}$$

- Hashimoto-Shirahata test ([12]) with test statistic

$$C_n = \binom{n}{4}^{-1} \sum_{i<j<k<l} h(X_i, X_j, X_k, X_l); \tag{16}$$

  where

$$\begin{aligned}
&h(X_i, X_j, X_k, X_l) = \frac{1}{36}\Big((X_i - X_j)^2 + (X_i - X_k)^2 + (X_i - X_l)^2 \\
&+ (X_j - X_k)^2 + (X_j - X_l)^2 + (X_k - X_l)^2\Big) \\
&- \frac{1}{6}\Big((\max(X_i, X_j) - \max(X_k, X_l))(\min(X_j, X_j) - \min(X_k, X_l) \\
&+ (\max(X_i, X_k) - \max(X_j, X_l))(\min(X_i, X_k) - \min(X_j, X_l) \\
&+ (\max(X_i, X_l) - \max(X_j, X_k))(\min(X_i, X_l) - \min(X_j, X_k))\Big).
\end{aligned}$$

We calculate local Bahadur efficiencies of proposed tests and the competitor ones against the following alternatives:

- a power function distribution with density function

$$g_1(x,\theta) = (\theta+1)x^{\theta},\ x \in (0,1),\ \theta > 0; \tag{17}$$

- a distribution with density

$$g_2(x,\theta) = 1 + \theta(2x-1)\ x \in (0,1),\ \theta > 0; \tag{18}$$

- a mixture of uniform and power function distributions

$$g_3(x,\theta) = 1 - \theta + \beta\theta x^{\beta-1},\ x \in (0,1); \tag{19}$$

- a second Ley-Paindaveine alternative with density function

$$g_4(x,\theta) = 1 - \theta\pi \sin \pi x,\ x \in (0,1)\ \theta \in [0, \pi^{-1}]. \tag{20}$$

It can be shown that those alternatives satisfy regularity conditions from [22].

In order to calculate local Bahadur efficiencies we use the results for exact Bahadur slopes from [18] and [8] for (12)-(15). The large deviations functions are respectively

$$\begin{aligned}
f_D(a) &= -2a^2 + o(a^2),\ a \to 0,\\
f_A(a) &= -a + o(a^2),\ a \to 0,\\
f_Q(a) &= -\frac{5}{2}a^2 + o(a^2),\ a \to 0.
\end{aligned}$$

Since for Hashimoto-Shirahata test the exact Bahadur slope is not derived yet, we do it here.

Notice that C_n is an $U-$ statistic with kernel $h(X_1, X_2, X_3, X_4)$. It is easy to show that the kernel is weakly degenerate. The projection of the kernel on X_1 and X_2 is equal to

$$\begin{aligned}
h^*(s_1,s_2) = E(h(X_1,X_2,X_3,X_4)|X_1 = s_1, X_2 = s_2) &= \frac{s_1^2 s_2 + s_2^2 s_1}{6}\\
+ \frac{\min(s_1,s_2)}{18} - \frac{2s_1 s_2}{9} - \frac{s_1^2 s_2^2}{6}
\end{aligned}$$

According to (8) we find that the smallest λ satisfies the integral equation

$$x(s_1) = \lambda \int_0^1 h^*(s_1,s_2)x(s_2)ds_2,\ \int_0^1 x(s_2)ds_2 = 0,$$

or equivalently

$$x'''(s_1) = -\frac{\lambda}{18}x'(s_1),\ x(0) = x(1) = 0,\ \int_0^1 x(s_2)ds_2 = 0.$$

This differential equation has a solution if and only if λ is the solution of the following equation

$$\sin\frac{\sqrt{\lambda}}{6\sqrt{2}}\Big(\sqrt{\lambda}\cos\frac{\sqrt{\lambda}}{6\sqrt{2}} - 6\sqrt{2}\sin\frac{\sqrt{\lambda}}{6\sqrt{2}}\Big) = 0.$$

The smallest positive solution is $\lambda_0 = 72\pi^2$.

In Table 1 we present the local Bahadur efficiencies of considered tests for alternatives (17)-(20). The bolded numbers represent the cases where our tests outperformed, in Bahadur sense, the competitor ones.

Table 1: Local Bahadur efficiency

Alternative	A	D	$T^{(2)}$	Q^c	C	$T^{(1)}$
g_1	0.40	0.54	**0.81**	0.14	0.37	**0.73**
g_2	0.5	0.75	**0.95**	0	0.66	**0.98**
$g_3(3)$	0.48	0.74	**0.82**	0.06	0.63	**0.94**
g_4	0.49	0.81	**0.96**	0	0.76	**1**

Table 2: Local Bahadur efficiency for location alternatives

Statistics	Gaussian	Cauchy
A_n^2	0.96	0.66
D_n	0.64	0.81
Q_n^c	0	0
C_n	0.49	1
$T_n^{(1)}$	0.955	0.76
$T_n^{(2)}$	0.87	0.72

In Table 2 we present the local Bahadur efficiency of our tests applied to normal and Cauchy null distribution and corresponding location alternatives. We use notation from ([18], Chapter 2).

It is interesting to note that Hashimoto-Shirahata is locally optimal for location alternative of Cauchy distribution.

4 Local optimality

In this section we present a class of locally optimal alternatives for each test. Following theorem follows from Theorem 5 from [22].

Theorem 4.1 *Let $g(x;\theta)$ be the density from $\mathcal{G}$. Then, for small θ, alternative densities*

$$g(x;\theta) = 1 + \theta(Cf_{0,m}(x)),\ x \in [0,1],\ C > 0,$$

where $f_{0,m}(x)$ is the eigenfunction that corresponds to first eigenvalue of the integral operator (2)*, is a class of locally optimal alternatives for test based on T_n^m.*

Applying this theorem we obtain that for $T^{(1)}$ some of the locally optimal densities are

$$g_{(1),1}(x,\theta) = 1 + \theta\cos\pi x,\quad x \in [0,1], \tag{21}$$

$$g_{(1),2}(x,\theta) = \frac{3(2-\theta)\theta}{2\arctan\sqrt{\frac{3\theta}{2-\theta}}}\frac{1}{1-\theta\cos\pi x}, x \in [0,1]. \tag{22}$$

Since the locally optimal densities for $T^{(2)}$ are to complicated to display we present them in Figures1 and 2.

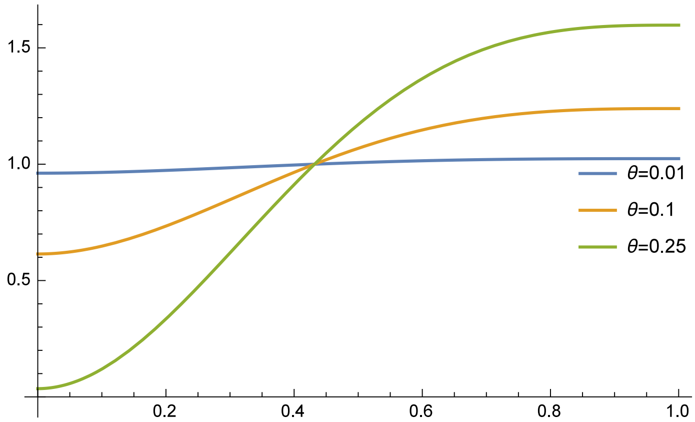

Figure 1: Locally optimal alternatives $g_{(2),1}(x, \theta)$

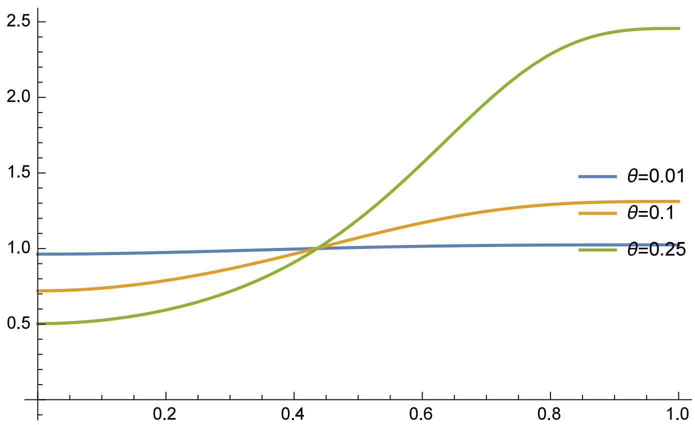

Figure 2: Locally optimal alternatives $g_{(2),2}(x, \theta)$

5 Power study and application

The powers of our and competitors tests are presented in Figures 3-6. The powers are calculated using Monte Carlo simulations with 10000 replicates at the level of significance 0.1.

From these figures we can conclude that for all considered values of parameter θ our tests outperform the competitor ones.

Goodness-of-fit tests for uniformity have application in time series analysis. We illustrate this with the following example.

Consider time series generated by process

$$x_t = \sin \frac{t}{3}\pi + z_t, \ t = 1...100. \tag{23}$$

We want to test that x_t is Gaussian white noise and there are no hidden pe-

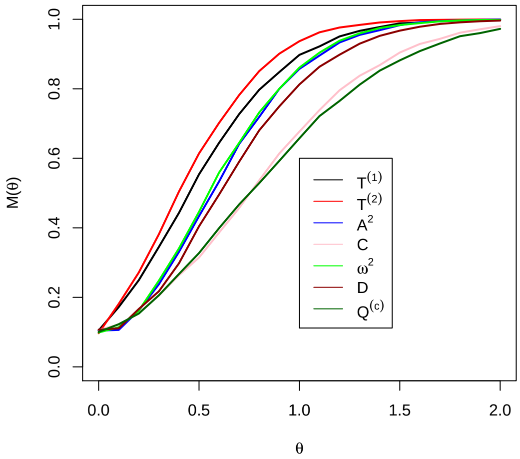

Figure 3: Powers of test for power alternatives

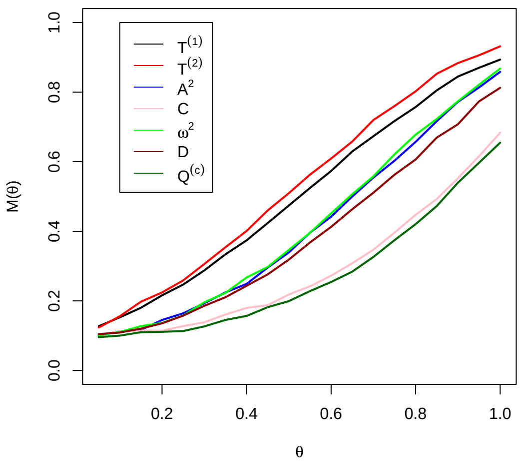

Figure 4: Powers of test for g_2 alternatives

riodicities. Using the results from [5] we have that if null hypothesis is true than

$$Y_k = \frac{\sum_{i=1}^{k} I(\omega_i)}{\sum_{i=1}^{q-1} I(\omega_i)}, \quad k = 1.., 48,$$

where $\omega_i = 2\pi\frac{i}{n}$ are Fourier frequencies and I is periodogram, are the ordered statistics from uniform $U[0,1]$ distribution. Therefore we may apply our test.

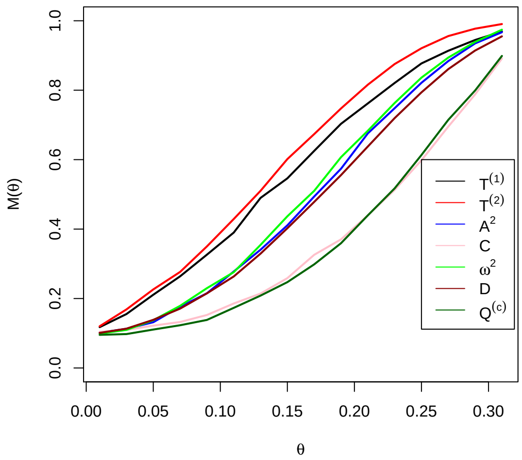

Figure 5: Powers of test for Ley-Paindaveine alternatives

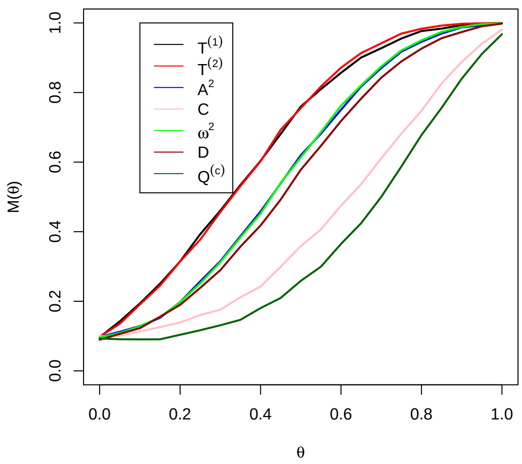

Figure 6: Powers of test for mixture with power alternatives

We obtain the p-value less than 0.05 and therefore we may confirm the existence of hidden periodicities.

6 Conclusion

In this paper we proposed new goodness-of-fit tests for uniform and other continuous distributions based on moment characterizations. We found their Bahadur efficiencies under close alternatives and showed that in almost all considered cases our tests were more efficient than common competitor tests. We calculated the powers of our tests for small sample sizes and noticed that in most cases our tests were more powerful. Finally we presented one application of our tests in time series analysis.

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